

CHARACTERIZATION THEOREM ON LOSSES IN $GI^X/GI^Y/1/n$ QUEUES

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ABSTRACT. In this paper, we prove a characterization theorem on the number of losses during a busy period in $GI^X/GI^Y/1/n$ queueing systems, in which the interarrival time distribution belongs to the class NWUE.

1. INTRODUCTION

There are several papers that study the properties of losses from queues during their busy periods. In [1] and then in [8] and [10], it was proved that in $M/GI/1/n$ queueing systems, in which the expectations of interarrival and service times are equal, the expected number of losses during their busy periods is equal to 1 for all n . In [3], this result was extended for $M^X/GI/1/n$ queues. In [2] and [4] some stochastic inequalities connecting the number of losses during busy periods in $M/GI/1/n$ and $GI/M/1/n$ queues and the number of offspring in Galton-Watson branching processes were obtained, and in [10], for $GI/GI/1/n$ queueing systems in which interarrival time distribution belongs to the class NBUE or NWUE, simple inequalities for the expected number of losses during a busy period were obtained. Peköz, Righter and Xia [7] gave a characterization of the number of losses during a busy period of $GI/M/1/n$ queueing systems. They proved that if the expected number of losses during a busy period is equal to 1 for all n , then arrivals must be Poisson.

In the present paper, we prove a characterization theorem for the expected number of losses during a busy period for the class of $GI^X/GI^Y/1/n$ queueing systems, in which interarrival time distribution belongs to the class NWUE.

Recall that the probability distribution function of a random variable ξ is said to belong to the class NBUE if for any $x \geq 0$ the inequality $E\{\xi - x | \xi > x\} \leq E\xi$ holds. If the opposite inequality holds, i.e. $E\{\xi - x | \xi > x\} \geq E\xi$, then the probability distribution function of a random variable ξ is said to belong to the class NWUE.

The queueing system $GI^X/GI^Y/1/n$, where the symbols X and Y denote an arrival batch and, respectively, service batch, is characterized by parameter n and four (control) sequences $\{\tau_i, X_i, \chi_i, Y_i\}$ of random variables ($i = 1, 2, \dots$), each of which consists of independently and identically distributed random variables, and these sequences are independent of each other (e.g. see [5]). Let $A(x) = P\{\tau_i \leq x\}$ denote the interarrival time probability distribution function, $a = \int_0^\infty x dA(x) < \infty$, and let $B(x) = P\{\chi_i \leq x\}$ denote the service time probability distribution function,

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$b = \int_0^\infty x dB(x) < \infty$. The random variables X_1, X_2, \dots denote consecutive masses of arriving units or, in known terminology, their batch sizes; but here they are assumed to be positive real-valued random variables rather than integer-valued. In turn, the random variables Y_1, Y_2, \dots denote consecutive service masses or service batches, which are also assumed to be positive real-valued random variables. The i th service batch Y_i characterizes the quantity that can be processed during the i th service time given that the necessary quantity is available in the system immediately before the i th service. Both X_1 and Y_1 are assumed to have finite expectations. In addition, the capacity of the system n is assumed to be a positive real number in general.

The queueing systems with real-valued batch arrival and service, as well as with real-valued capacity are not traditional. They can be motivated, however, in industrial applications, where units can be lorries with sand or soil arriving in and departing from a storage station. In usual queueing formulations, where the random variables X_i and Y_i are integer-valued, they are characterized as batches of arrived and served customers. The main result of the present paper, Theorem 1.1, is new even in the particular case when $X_i = Y_i = 1$ and n is integer.

Theorem 1.1. *Let M_L denote the total mass lost during a busy period. Assume that the probability distribution function $A(x)$ belongs to the class NWUE. Then the equality $\mathbf{E}M_L = \mathbf{E}X_1$ holds for all positive real n if and only if arrivals are Poisson, the random variable Y_1 takes a single value d , the probability distribution function of X_1 is lattice with span d , and $\mathbf{E}X_1 = \frac{ad}{b}$.*

This theorem is true for full and partial rejection policies, work-conserving disciplines and can be adapted to different models considered, for instance, in [10]. The characterization theorem is a necessary and sufficient condition that includes the case of the Poisson arrivals. However, if arrivals are not Poisson but belong to the practically important class NWUE, then in the case where $\frac{1}{a}\mathbf{E}X_1 \geq \frac{1}{b}\mathbf{E}Y_1$, i.e. in the case where the total mass of arrivals per unit time is not smaller than the total mass of service per unit time, we have the simple inequality given by Lemma 2.1 that is used to prove Theorem 1.1.

2. PROOF OF THE MAIN RESULT

We start from the following lemma.

Lemma 2.1. *Assume that the length of a busy period has a finite mean, and that the probability distribution function $A(x)$ belongs to the class NWUE. If $\frac{1}{a}\mathbf{E}X_1 \geq \frac{1}{b}\mathbf{E}Y_1$ and $\mathbf{P}\{X_1 \leq n\} > 0$, then for any nontrivial random variable Y_1 (i.e. taking at least two positive values) we have $\mathbf{E}M_L > \mathbf{E}X_1$.*

Proof. Let N_A denote the total number of arrivals during a busy cycle (that is, total number of arrivals during a busy period plus the unit that starts the busy period), and let N_S denote the total number of service completions during the busy period. Denoting by M_A the total mass of arrivals during a busy cycle, and, respectively, by M_S the total mass of served units during a busy period. Using Wald's identity, we have:

$$(2.1) \quad \mathbf{E}M_A = \mathbf{E}X_1 \mathbf{E}N_A,$$

$$(2.2) \quad a\mathbf{E}N_A = b\mathbf{E}N_S + \mathbf{E}I,$$

where I in Equation (2.2) denotes the length of idle time. Since $A(x)$ belongs to the class NWUE, then $EI \geq a$ (see [9], p.482). Hence, from (2.2) we have

$$(2.3) \quad aEN_A - a \geq bEN_S.$$

Note, that for M_S we cannot use Wald's identity directly in order to show that $EM_S \leq EY_1EN_S$. In order to prove this inequality, we introduce the sequence of random variables S_1, S_2, \dots that characterizes *real* masses of service or, in other words, *real* batch sizes of service satisfying the properties $E\{S_j|N_S = j\} \leq EY_1$ while $E\{S_i|N_S = j\} = EY_1$ for $i < j$ ($1 \leq i < j$). Apparently S_1, S_2, \dots are *not* independent random variables. So, additional properties of the sequence S_1, S_2, \dots are needed in order to establish the inequality for EM_S .

Let m_i denote the workload of the system immediately before the service of the i th unit starts. Then, given $\{m_1 = x_1, m_2 = x_2, \dots\}$ ($x_i \leq n$ for all i), the sequence S_1, S_2, \dots is conditionally independent. Hence, under the condition $\{m_1 = x_1, m_2 = x_2, \dots\}$ for the sequence of conditionally independent random variables S_1, S_2, \dots one can use the following theorem by Kolmogorov and Prohorov [6].

Lemma 2.2. (*Kolmogorov and Prohorov [6].*) *Let ξ_1, ξ_2, \dots be independent random variables, and let ν be an integer random variable such that the event $\{\nu = k\}$ is independent of $\xi_{k+1}, \xi_{k+2}, \dots$. Assume that $E\xi_k = v_k$, $E|\xi_k| = u_k$ and the series $\sum_{k=1}^{\infty} P\{\nu \geq k\}u_k$ converges. Then,*

$$E \sum_{i=1}^{\nu} \xi_i = \sum_{k=1}^{\infty} P\{\nu \geq k\} \sum_{i=1}^k v_i.$$

Note, that the condition $\sum_{k=1}^{\infty} P\{\nu \geq k\}u_k < \infty$ of Lemma 2.2 is satisfied, because $EN_S < \infty$ and $ES_k \leq n$ for all k . Hence, by the total expectation formula we have $ES_j \leq EY_1$, and consequently by Lemma 2.2 and the total expectation formula we arrive at $EM_S \leq EN_SEY_1$. We show below, that in fact we have the strict inequality $EM_S < EN_SEY_1$.

Indeed, the fact that the probability distribution function $A(x)$ belongs to the class NWUE implies that $\bar{A}(x) = 1 - A(x) > 0$ for any x . Hence, taking into account that Y_1 takes at least two different positive values, one can conclude that there exists the value j_0 such that $E\{S_{j_0}|N_S = j_0\} < EY_1$, and consequently $ES_{j_0} < EY_1$. This implies

$$(2.4) \quad EM_S < EN_SEY_1.$$

Now (2.1), (2.3) and (2.4) and the equality $EM_L = EM_A - EM_S$ allows us to obtain the inequality $EM_L > EX_1$. \square

Remark 2.3. In the formulation of the Lemma 2.1 we assumed that the length of a busy period has a finite mean. This assumption is technically important in order to use Wald's identity. Note, that the assumption that the interarrival time distribution belongs to the class NWUE implies $\bar{A}(x) = 1 - A(x) > 0$ for any x , which consequently enables us to conclude that a busy period always exists (i.e. finite) with probability 1. If the expectation of the busy period length is infinite, then the expected number of losses during a busy period is infinite as well, and hence the statement of Lemma 2.1 in this case remains true.

Proof of Theorem 1.1. Note first that if $\frac{1}{a}EX_1 < \frac{1}{b}EY_1$, then EM_L vanishes as $n \rightarrow \infty$ due to the law of large numbers. Hence, the only case $\frac{1}{a}EX_1 \geq \frac{1}{b}EY_1$ is available, and this is the assumption in Lemma 2.1.

Hence, the problem reduces to a minimization problem for EM_L in the set of the possible values. More specifically, the problem is to find the infimum of EM_L subject to the constraints given by (2.1), (2.3), (2.4) and the inequality $\frac{1}{a}EX_1 \geq \frac{1}{b}EY_1$. Then, the statement of this theorem follows if and only if along with (2.1) we also have $aEN_A - a = bEN_S$, $EM_S = EN_SEY_1$ and $\frac{1}{a}EX_1 = \frac{1}{b}EY_1$. The first equality follows if and only if arrivals are Poisson, and the second one follows if and only if Y_1 takes a single value d , and the probability distribution function of X_1 is lattice with span d . Then, the system of three equations together with the equality $EX_1 = \frac{ad}{b}$ (which in turn is a consequence of $\frac{1}{a}EX_1 = \frac{1}{b}EY_1$) yields the desired result $EM_L = EX_1$.

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